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RELAXATION OF TURBULENT STRESS

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Equations for the pulsational components of the velocity and temperature yield relaxational formulas for the turbulent stress and the heat flux.

1. Relaxation of Turbulent Stress

Attention was first drawn to the analogy between a turbulent flow of Newtonian fluid and a laminar flow of non-Newtonian fluid in [1]. There have been a number of works in which turbulence phenomena are identified with viscoelasticity phenomena [2-4]. In [5], attention turned to the possibility of making use of the pulsational components of the velocity to close the equations for the mean velocities.

From these equations, by well-known means, it is possible to write relaxational formulas for the turbulent stress. To this end, consider the general equations of motion of an incompressible liquid in the absence of external bulk forces

$$\frac{\rho\partial u_i}{\partial t} = -\frac{\partial}{\partial x_k}(p_{ki} + \rho u_i u_k), \tag{1}$$

$$\frac{\partial u_k}{\partial x_k} = 0,\ i = 1,\ 2,\ 3,\ k = 1,\ 2,\ 3. \tag{2}$$

According to Reynolds' concepts, the hydrodynamic quantities are separated into mean and pulsational components

$$u_i = \bar{u}_i + u_i',\ u_k = \bar{u}_k + u_k',\ p_{ki} = \bar{p}_{ki} + p_{ki}'. \tag{3}$$

Now the following equations for the mean and pulsational velocities may be derived from Eq. (1) [6]

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$$\frac{\rho\partial \bar{u}_i}{\partial t}=-\frac{\partial}{\partial x_k}\left(\bar{p}_{ki}+\rho\bar{u}_i\bar{u}_k+\rho\overline{u_i' u_k'}\right), \tag{4}$$

$$\frac{\rho\partial u_i'}{\partial t}=-\frac{\partial}{\partial x_k}\left[p_{ki}'+\rho\left(\bar{u}_i u_k'+u_i'\bar{u}_k\right)+\rho\left(u_i' u_k'-\overline{u_i' u_k'}\right)\right], \tag{5}$$

$$\frac{\rho\partial u_j'}{\partial t}=-\frac{\partial}{\partial x_k}\left[p_{kj}'+\rho\left(\bar{u}_j u_k'+u_j'\bar{u}_k\right)+\rho\left(u_j' u_k'-\overline{u_j' u_k'}\right)\right]. \tag{6}$$

From Eqs. (5) and (6), an expression is obtained for the components of the turbulent-stress tensor $\overline{u_i' u_j'}$, bearing in mind the obvious formula

$$\frac{\partial u_i' u_j'}{\partial t}=u_i'\frac{\partial u_j'}{\partial t}+u_j'\frac{\partial u_i'}{\partial t}.$$

Multiplying Eq. (5) by u_j' and Eq. (6) by u_i' and adding them yields the result

$$\frac{\rho d u_i' u_j'}{dt}=-u_j'\frac{\partial}{\partial x_k}\left[p_{ki}'+\rho\bar{u}_i u_k'+\rho\left(u_i' u_k'-\overline{u_i' u_k'}\right)\right]-u_i'\frac{\partial}{\partial x_k}\left[p_{kj}'+\rho\bar{u}_j u_k'+\rho\left(u_j' u_k'-\overline{u_j' u_k'}\right)\right]. \tag{7}$$

The convective-product operator has been introduced here

$$\frac{d u_i' u_j'}{dt}=\frac{\partial u_i' u_j'}{\partial t}+\bar{u}_k\frac{\partial u_i' u_j'}{\partial x_k}.$$

All the terms in Eq. (7) are now averaged, using the Reynolds equations

$$\overline{u_i'}=\overline{u_k'}=0;\quad \overline{\overline{u_i' u_k'}}=\overline{u_i' u_k'};\quad \bar{\bar{u}}_k=\bar{u}_k \tag{8}$$

and adopting the hypothesis that

$$\overline{u_j'\frac{\partial \overline{u_i' u_k'}}{\partial x_k}}=0;\quad \overline{u_i' u_k'\frac{\partial u_j'}{\partial x_k}}=0. \tag{9}$$

Then Eq. (7) takes the form

$$\frac{\rho d\overline{u_i' u_j'}}{dt}=-\left\{\frac{\partial}{\partial x_k}\overline{\left[u_j'\left(p_{ki}'+\rho u_i' u_k'\right)\right]}+\frac{\partial}{\partial x_k}\overline{\left[u_i'\left(p_{kj}'+\rho u_j' u_k'\right)\right]}\right\}$$
$$+\overline{p_{ki}'\frac{\partial u_j'}{\partial x_k}}+\overline{p_{kj}'\frac{\partial u_i'}{\partial x_k}}-\left[\rho\overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k}+\rho\overline{u_i' u_k'}\frac{\partial \bar{u}_j}{\partial x_k}\right]. \tag{10}$$

Integration over an arbitrary volume V is now performed in Eq. (10), and in the first two terms on the right of the resulting expression the Ostrogradskii–Gauss theorem is used in the following form

$$\int_V\left\{\frac{\partial}{\partial x_k}\overline{\left[u_j'\left(p_{ki}'+\rho u_i' u_k'\right)\right]}+\frac{\partial}{\partial x_k}\overline{\left[u_i'\left(p_{kj}'+\rho u_j' u_k'\right)\right]}\right\}dV=\int_S n_k\left\{\frac{\partial}{\partial x_k}\overline{\left[u_j'\left(p_{ki}'+\rho u_i' u_k'\right)\right]}+\frac{\partial}{\partial x_k}\overline{\left[u_i'\left(p_{kj}'+\rho u_j' u_k'\right)\right]}\right\}dS.$$

Here n_k are the directional cosines of the normal to the surface S. The surface integral expresses the energy transfer with respect to motion through the surface S and, according to the discussion of Reynolds [6], is equal to zero. Then, the following relation replaces Eq. (10)

$$\int_V\frac{\rho d\overline{u_i' u_j'}}{dt}\,dV=\int_V\left(\overline{p_{ki}'\frac{\partial u_j'}{\partial x_k}}+\overline{p_{kj}'\frac{\partial u_i'}{\partial x_k}}\right)dV-\int_V\rho\left(\overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k}+\overline{u_i' u_k'}\frac{\partial \bar{u}_j}{\partial x_k}\right)dV.$$

But since the elementary volume V is arbitrary, this equation may be rewritten in the form

$$\frac{\rho d\overline{u_i' u_j'}}{dt}=\overline{p_{ki}'\frac{\partial u_j'}{\partial x_k}}+\overline{p_{kj}'\frac{\partial u_i'}{\partial x_k}}-\rho\left(\overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k}+\overline{u_i' u_k'}\frac{\partial \bar{u}_j}{\partial x_k}\right). \tag{11}$$

Let $i=j$; then Eq. (11) gives

$$\frac{d\bar{E}'}{dt}=\overline{p_{ki}'\frac{\partial u_i'}{\partial x_k}}-\rho\overline{u_i' u_k'}\frac{\partial \bar{u}_i}{\partial x_k}, \tag{12}$$

where $\bar{E}' = 1/2\ \overline{u_i'^2}$ is the kinetic energy of motion along the trajectory of mean motion [6]. Reynolds calls this the "resolving" equation, and has attempted to calculate the critical number Re from it.

For the mean values, by analogy with the usual representation, the following formulas are introduced

$$\bar{p}_{ij} = \begin{cases} -\mu_{ij}\left(\dfrac{\partial \bar{u}_i}{\partial x_j} + \dfrac{\partial \bar{u}_j}{\partial x_i}\right), & i \neq j, \\ \bar{p} - 2\mu_{ii}\left(\dfrac{\partial \bar{u}_i}{\partial x_i}\right), & i = j. \end{cases} \tag{13}$$

Using Eq. (13), the mean-deformation-rate tensor can be expressed in terms of the Maxwellian relaxation time

$$\begin{gathered} \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} = -\frac{\bar{p}_{ij}}{\mu_{ij}} = -\frac{1}{\tau_{ij}}, \quad i \neq j, \\ 2\,\frac{\partial \bar{u}_i}{\partial x_i} = \frac{\bar{p}_{ij} - \bar{p}}{-\mu_{ii}} = -\frac{1}{\tau_{ii}}. \end{gathered} \tag{14}$$

Introducing the usual formulas for the turbulent stress

$$\sigma_{ij} = -\rho\overline{u_i' u_j'},$$

Eq. (11) can be rewritten in the form

$$-\frac{d\sigma_{ij}}{dt} = \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \sigma_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + \overline{p_{ki}'\frac{\partial u_j'}{\partial x_k}} + \overline{p_{kj}'\frac{\partial u_i'}{\partial x_k}}. \tag{15}$$

The well-known formulas for deformational motion are used

$$\begin{aligned} \frac{\partial \bar{u}_i}{\partial x_k} &= \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} - \frac{\partial \bar{u}_k}{\partial x_i}\right), \\ \frac{\partial \bar{u}_j}{\partial x_k} &= \frac{1}{2}\left(\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_j}\right) + \frac{1}{2}\left(\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial \bar{u}_k}{\partial x_j}\right). \end{aligned} \tag{16}$$

Taking these formulas into account, Eq. (15) may be divided into two:

$$-\frac{d\sigma_{ij}}{dt} = \frac{\sigma_{jk}}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_i}\right) + \frac{\sigma_{ik}}{2}\left(\frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial \bar{u}_k}{\partial x_j}\right), \tag{17}$$

$$\overline{p_{ki}'\frac{\partial u_j'}{\partial x_k}} + \overline{p_{kj}'\frac{\partial u_i'}{\partial x_k}} = -\frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_k} - \frac{\partial \bar{u}_k}{\partial x_i}\right) - \frac{1}{2}\left(\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial \bar{u}_k}{\partial x_j}\right). \tag{18}$$

Equation (18) expresses the dissipation of relative motion through vorticity of the mean motion.

If the mean velocities are eliminated in Eq. (17), in accordance with Eq. (14), the result is

$$-\frac{d\sigma_{ij}}{dt} + \frac{1}{2}\left(\frac{1}{\tau_{ii}} + \frac{1}{\tau_{jj}}\right)\sigma_{ij} + \frac{\sigma_{ii} + \sigma_{jj}}{2\,\tau_{ij}} + \frac{\sigma_{ik}}{2\tau_{jk}} + \frac{\sigma_{jk}}{2\tau_{ik}} = 0, \tag{19}$$

$$i, j = 1, 2, 3;\ k \neq i,\ k \neq j,$$

$$-\frac{d\sigma_{ii}}{dt} + \sum_{k=1}^{3}\frac{\sigma_{ik}}{\tau_{ik}} = 0. \tag{20}$$

The expressions obtained are relaxational formulas for the turbulent stress.

For the purposes of subsequent discussion, Eq. (4) is rewritten in the form

$$\rho\,\frac{d\bar{u}_i}{dt} = -\frac{\partial}{\partial x_k}\left(\bar{p}_{ki} - \sigma_{ki}\right). \tag{21}$$

The turbulent stress is eliminated by a means used in deriving heat-conduction equations of hyperbolic type [7]. Both sides of Eq. (21) are differentiated with respect to time and the resulting relation is premultiplied by τ_{ii}, to give

$$\tau_{ii}\rho \frac{d^2\bar{u}_i}{dt^2} = -\tau_{ii} \frac{\partial}{\partial x_k}\left[\frac{d}{dt}(\bar{p}_{ki} - \sigma_{ki})\right]. \tag{22}$$

Adding Eqs. (21) and (22) leads to the relation

$$\tau_{ii}\rho \frac{d^2\bar{u}_i}{dt^2} + \rho \frac{d\bar{u}_i}{dt} = -\frac{\partial}{\partial x_k}\left[(\bar{p}_{ki} - \sigma_{ki}) + \tau_{ii}\frac{d}{dt}(\bar{p}_{ki} - \sigma_{ki})\right]. \tag{23}$$

To close this equation, the following hypothesis is introduced

$$(\bar{p}_{ki} - \sigma_{ki}) + \tau_{ii}\frac{d}{dt}(\bar{p}_{ki} - \sigma_{ki}) = \begin{cases} -\mu_{ki}\left(\frac{\partial \bar{u}_k}{\partial x_i} + \frac{\partial \bar{u}_i}{\partial x_k}\right), & k \neq i, \\ \bar{p} - 2\mu_{ii}\left(\frac{\partial \bar{u}_i}{\partial x_i}\right), & k = i. \end{cases} \tag{24}$$

This is analogous to the Maxwellian hypothesis for viscoelastic stress. Now Eq. (23) is rewritten in the form:

$$\tau_{ii}\rho \frac{d^2\bar{u}_i}{dt^2} + \rho \frac{d\bar{u}_i}{dt} = -\frac{\partial \bar{p}}{\partial x_i} + \mu_{ki}\frac{\partial^2 \bar{u}_i}{\partial x_k^2} + \frac{\partial}{\partial x_i}\left(\mu_{ki}\frac{\partial \bar{u}_k}{\partial x_k}\right),$$

$$i,\ k = 1,\ 2,\ 3;\ \mu_{ki} = \mu_{ik}. \tag{25}$$

If the velocities of mean motion are found from Eq. (25), then σ_{ij} and $\bar{p}_{ki}$ may be found from Eqs. (19), (20), and (24).

The formula for the second convective product takes the form

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + u_k^2 \frac{\partial^2}{\partial x_k^2} + \frac{d\bar{u}_k}{dt}\frac{\partial}{\partial x_k} + 2\bar{u}_k \frac{\partial^2}{\partial t \partial x} + 2\bar{u}_1\bar{u}_2\frac{\partial^2}{\partial x_1 \partial x_2} + 2\bar{u}_1\bar{u}_3\frac{\partial^2}{\partial x_1 \partial x_3} + 2\bar{u}_2\bar{u}_3\frac{\partial^2}{\partial x_2 \partial x_3}. \tag{26}$$

In the simplest case, when $\tau_{ij} = \tau$, $\mu_{ki} = \mu$, Eq. (25) takes the simpler form

$$\tau \frac{d^2\bar{u}_i}{dt^2} + \frac{d\bar{u}_i}{dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\nabla^2\bar{u}_i. \tag{27}$$

2. Relaxation of Heat Flux

Consider the heat-balance equation in a turbulent flow of incompressible liquid, disregarding energy dissipation

$$c_0 \frac{dT}{dt} = -\frac{\partial q_k}{\partial x_k}. \tag{28}$$

Here c_0 is the product of the specific heat and the density. Separating the quantities appearing in Eq. (28) into mean and pulsational components

$$T = \bar{T} + T',\ u_k = \bar{u}_k + u_k',\ q_k = \bar{q}_k + q_k' \tag{29}$$

and performing the usual averaging operation for all the terms in Eq. (28) leads to the result

$$c_0 \frac{d\bar{T}}{dt} = -\frac{\partial}{\partial x_k}(\bar{q}_k + c_0\overline{u_k'T'}). \tag{30}$$

Subtracting Eq. (30) from Eq. (28) gives

$$c_0 \frac{dT'}{dt} = -\frac{\partial}{\partial x_k}[q_k' + c_0 u_k'\bar{T} + c_0(u_k'T' - \overline{u_k'T'})]. \tag{31}$$

By the above-outlined means, the following equation may be obtained from Eqs. (31) and (5):

$$\overline{\frac{du'T'}{dt}} = \frac{1}{\rho}\overline{p_{ki}'\frac{\partial T'}{\partial x_k}} + \frac{1}{c_0}\overline{q_k'\frac{\partial u_i'}{\partial x_k}} - \left(\overline{T'u_k'}\frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_i'u_k'}\frac{\partial \bar{T}}{\partial x_k}\right). \tag{32}$$

It has been assumed that

$$\overline{u_i'\frac{\partial \overline{u_k'T'}}{\partial x_k}} = 0,\ \overline{u_k'T'\frac{\partial u_i'}{\partial x_k}} = 0.$$

If Eq. (16) is used, Eq. (32) may be replaced by two relations:

$$\frac{d\overline{u_i'T'}}{dt}+\frac{\overline{u_k'T'}}{2}\left(\frac{\partial\bar{u}_i}{\partial x_k}+\frac{\partial\bar{u}_k}{\partial x_i}\right)=0, \tag{33}$$

$$\frac{1}{c_0}\overline{q_k'\frac{\partial u_i'}{\partial x_k}}+\frac{1}{\rho}\overline{p_{ki}'\frac{\partial T'}{\partial x_k}}=\overline{u_i'u_k'}\frac{\partial\bar{T}}{\partial x_k}+\frac{\overline{u_k'T'}}{2}\left(\frac{\partial\bar{u}_i}{\partial x_k}-\frac{\partial\bar{u}_k}{\partial x_i}\right). \tag{34}$$

In (33), the mean values may be expressed in terms of the relaxation-time spectrum in Eq. (14), when

$$\frac{d\overline{u_i'T'}}{dt}-\frac{\overline{u_k'T'}}{2\tau_{ik}}=0. \tag{35}$$

After differentiation with respect to the time, Eq. (30) is rewritten in the form

$$\tau_{ii}c_0\frac{d^2\bar{T}}{dt^2}=-\tau_{ii}\frac{\partial}{\partial x_k}\left[\frac{d}{dt}(\bar{q}_k+c_0\overline{u_k'T'})\right].$$

The resulting relation is combined with Eq. (30)

$$\tau_{ii}c_0\frac{d^2\bar{T}}{dt^2}+c_0\frac{d\bar{T}}{dt}=-\frac{\partial}{\partial x_k}\left[(\bar{q}_k+c_0\overline{u_k'T'})+\tau_{ii}\frac{d}{dt}(\bar{q}_k+c_0\overline{u_k'T'})\right]$$

and the following hypothesis is adopted

$$(\bar{q}_k+c_0\overline{u_k'T'})+\tau_{ii}\frac{d}{dt}(\bar{q}_k+c_0\overline{u_k'T'})=-\lambda_k\frac{\partial\bar{T}}{\partial x_k}. \tag{36}$$

The result obtained is

$$\tau_{ii}\frac{d^2\bar{T}}{dt^2}+\frac{d\bar{T}}{dt}=k_i\frac{\partial^2\bar{T}}{\partial x_i^2}. \tag{37}$$

If Eq. (37) is solved, then $\bar{q}_k$ and $\overline{u_i'T'}$ may be determined from Eqs. (35) and (36).

3. Problem of Accelerating Flow Close to a Plane Wall

Suppose that a plane wall, previously at rest, suddenly begins to move in its own plane at a constant velocity U_0. For such a flow, $\bar{u}_1=\bar{u}_1(x_2,\ t)$, $\bar{u}_2=0$, $\bar{u}_3=0$, and the pressure remains constant [8]. With these assumptions, Eq. (27) is simplified:

$$\tau\frac{\partial^2 U}{\partial t^2}+\frac{\partial U}{\partial t}=\nu\frac{\partial^2 U}{\partial y^2}. \tag{38}$$

Here $U(t,\ y)=\bar{u}_1(x_2,\ t)$. The initial conditions are

$$\begin{gathered}\text{when } t\leqslant 0 \quad U=0 \text{ for all } y,\\ \text{when } t>0 \quad U=U_0 \text{ for } y=0,\ U=0,\ y=\infty.\end{gathered} \tag{39}$$

Let $\sigma=\bar{p}_{12}-\sigma_{12}$ denote the difference between the mean and turbulent stresses. Then Eq. (24) may be rewritten for this problem in the form:

$$\sigma+\tau\frac{\partial\sigma}{\partial t}=-\mu\frac{\partial U}{\partial y}. \tag{40}$$

It is now expedient to convert to dimensionless values by means of the formulas

$$t=\tau\bar{t},\ y=\sqrt{\nu\tau}\,\bar{y},\ U=U_0\bar{U},\ \sigma=\sigma_m\bar{\sigma}.$$

It may readily be established that Eqs. (38) and (39) are equivalent to the system of equations

$$\frac{\partial U}{\partial y}=-\left(\frac{\partial\sigma}{\partial t}+\sigma\right),\ \frac{\partial U}{\partial t}=-\frac{\partial\sigma}{\partial y}, \tag{41}$$

in which, for convenience, the bars above the dimensionless quantities have been omitted.

This system of equations was studied by Gon'o in connection with the theory of the hyperbolic heat-conduction equation, and its results are outlined in detail in [7]. According to this, it is found that

TABLE 1. Numerical Values of Frictional Stress at Wall

t	$\frac{1}{\sqrt{\pi t}}$	$e^{-\frac{t}{2}} I_0\left(\frac{1}{2} t\right)$	t	$\frac{1}{\sqrt{\pi t}}$	$e^{-\frac{t}{2}} I_0\left(\frac{1}{2} t\right)$
0	∞	1	2,0	0,3989	0,4657
0,2	1,2616	0,9071	3,0	0,3257	0,3674
0,4	0,8921	0,8269	4,0	0,2821	0,3085
0,8	0,6308	0,6974	5,0	0,2523	0,2700
1,0	0,5642	0,6450	6,0	0,2303	0,2430
1,4	0,4768	0,5593	7,0	0,2132	0,2228
1,8	0,4205	0,4932	8,0	0,1995	0,2070

$$\sigma(t, y)=\begin{cases} e^{-\frac{t}{2}} I_0\left(\frac{\sqrt{t^2-y^2}}{2}\right), & t>y, \\ 0, & 0<t<y. \end{cases} \tag{42}$$

Consider the value of the stress σ at the wall:

$$\sigma(t, 0)=\sigma_0(t)=e^{-\frac{t}{2}} I_0\left(\frac{1}{2} t\right). \tag{43}$$

At sufficiently large t, using the asymptotic expansion for the Bessel function I_0, it may be shown that $\sigma_0(t)=1/\sqrt{\pi t}$. The theory of the Navier–Stokes equation leads exactly to this result [8], which is paradoxical when $t \to 0$, since $\sigma_0(t) \to \infty$. From Eq. (43), $\sigma_0(0)=1$. Hence, the dimensional stress σ at the initial instant is

$$\sigma=\sigma_m=\frac{\rho U_0 \sqrt{\nu}}{\sqrt{\tau}}.$$

It is simple to see that this "resting stress" must be overcome by the wall if the given mode of flow is to arise.

The idea here outlined leads to the assertion that the flow is initially turbulent but, at some point in time, it becomes laminar if the initial and boundary conditions are retained in the form in Eq. (39). The moment of transition is determined by the relaxational properties of the medium.

Table 1 gives numerical values of the frictional stress at the wall $\sigma_0(t)$. It follows from an analysis of these data that discrepancy between Eq. (43) and the results of the theory of the Navier–Stokes equations is observed at a sufficiently large interval of dimensionless time t.

NOTATION

u_i, component of hydrodynamic velocity; $\bar{u}_i$, mean component of velocity; u_i', pulsational component of velocity; p_{ki}, stress tensor; $\bar{p}_{ki}$, mean stress tensor; p_{ki}', pulsational stress tensor; ρ, density; T, temperature; q_k, heat-flux component; I_0, Bessel function of zeroth argument.

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